4: An Exponential-Time Algorithm

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Computing Equilibria in General Bimatrix Games.

Let (R, C) be a general bimatrix game, $R, C \in \mathbb{R}^{m \times n}$
Thus row player has m pure strategies, column player has n pure strategies, and $x \in \Delta_m$, $y \in \Delta_n$ are mixed strategies.
We use e_i to denote the column vector $[0 \dots 0 \underset{i}{1} 0 \dots 0]^T$

We will now give an exponential-time algorithm for computing equilibria in bimatrix games.
Recall: given $x \in \Delta_m$, $\text{Supp}(x) = \{i : x_i > 0\}$
Similarly for $y \in \Delta_n$.

Fix $S_R \subseteq [m]$, $S_C \subseteq [n]$ as subsets of pure strategies for the players. Consider the following LP:

$$
\begin{aligned}
P(S_R, S_C): \quad & \max \ 0 \\
\text{s.t.} \quad & x \in \Delta_m \\
& y \in \Delta_n \\
\forall i \notin S_R, \quad & x_i = 0 \\
\forall j \notin S_C, \quad & y_j = 0 \\
\forall i \in S_R, i' \in [m], \quad & (Ry)_i \geq (Ry)_{i'} \\
\forall j \in S_C, j' \in [n], \quad & (Cx)_j \geq (Cx)_{j'}
\end{aligned}
$$

A (the two constraints $x_i = 0$, $y_j = 0$)

B (the two best-response constraints)

So: (A) x is supported on S_R, y on S_C
(B) S_R is a subset of best-responses to y
S_C is a subset of best-responses to x

Theorem: (I) If (x^*, y^*) is a feasible soln to $P(S_R, S_C)$, then (x^*, y^*) is a NE of the game.
(II) If (x^*, y^*) is a NE, let $S_R^* = \text{supp}(x^*)$, $S_C^* = \text{supp}(y^*)$. (x^*, y^*) is a feasible soln. to $P(S_R^*, S_C^*)$.

Proof of (II): easy.

Note: (I) Every LP w/ rational coefficients has a rational soln. Hence, if utilities R, C are rational the game has a rational equilibrium.
(II) Let (x_1^*, y^*), (x_2^*, y^*) be two equilibria of (R, C). Then $\forall\, 0 \leq \lambda \leq 1$, $(\lambda x_1^* + (1-\lambda) x_2^*, y^*)$ is also an equilibrium.
(Q.) Prove (II) yourself.

Proof of theorem (I):

Claim 1: (x^*, y^*) is a NE iff
$$x^{*T} R y^* \geq e_i^T R y^* \quad \forall\, i \in [m]$$
$$y^{*T} C x^* \geq e_j^T C x^* \quad \forall\, j \in [n]$$

Proof: Easy.

Corollary: (x^*, y^*) is a NE iff
$$\text{supp}(x^*) \subseteq \arg\max_{i \in [m]} (Ry^*)_i$$
$$\& \ \text{supp}(y^*) \subseteq \arg\max_{j \in [n]} (Cx^*)_j$$

Proof of Theorem: By the constraints:
$$\text{supp}(x^*) \overset{A}{\subseteq} S_R \overset{B}{\subseteq} \arg\max_{i \in [m]} (Ry^*)_i$$
$$\text{supp}(y^*) \overset{A}{\subseteq} S_C \overset{B}{\subseteq} \arg\max_{j \in [n]} (Cx^*)_j$$

Hence, (x^*, y^*) is a NE by the corollary.

By Nash's Theorem, we know $\exists$ a NE, hence $\exists\, S_R, S_C$ for which $P(S_R, S_C)$ is feasible.
Algorithm enumerates over all possible $S_R \subseteq [m]$, $S_C \subseteq [n]$, solves $P(S_R, S_C)$ to check if feasible, takes time
$$\text{poly}(m, n, |R|, |C|)\ 2^{m+n}$$